15.7 Lecture 2: Spherical Coordinates

Jeremiah Southwick
(And Robert Vandermolen)

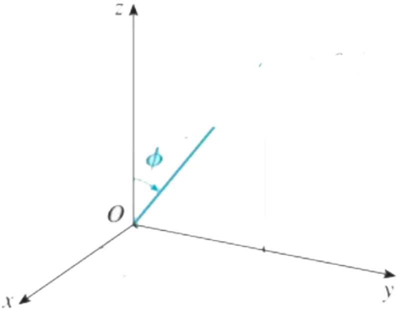
Spring 2019

Links

Robert's slides can be found here:
http://people.math.sc.edu/robertv/teaching.html
The 15.7 slides can be found here:
https://docs.google.com/presentation/d/1V_
CtHJvjz4-etPuIfNYLhpO8ohjmrU6nbetc9xTzqiU

Spherical Coordinates!

Sometimes the easiest conversion is into Spherical Coordinates, as follows



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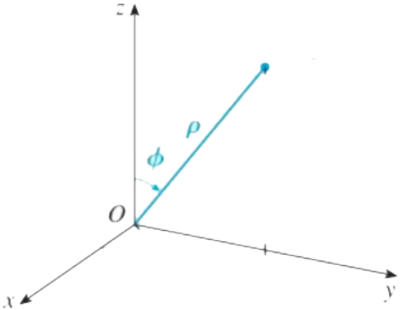

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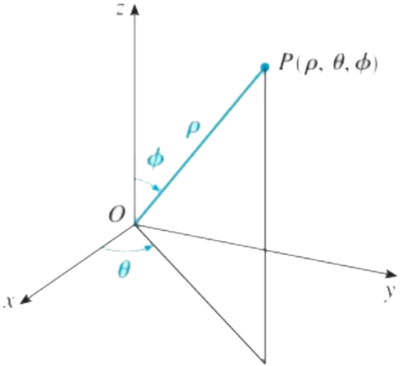
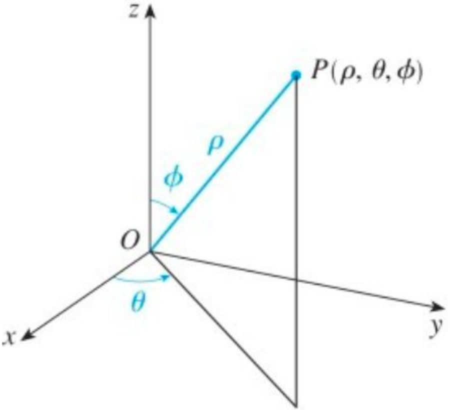


Spherical Coordinates!

Sometimes the easiest conversion is into Spherical Coordinates, as follows

Note, that ρ is again the length, so we have an easy formula relating it to x,y,z:

$$\rho^2 = x^2 + y^2 + z^2$$



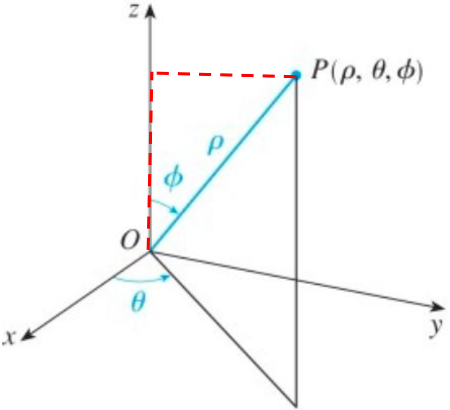
Spherical Coordinates!

Sometimes the easiest conversion is into Spherical Coordinates, as follows

$$z = \rho\cos(\phi)$$

Note, that ρ is again the length, so we have an easy formula relating it to x,y,z:

$$\rho^2 = x^2 + y^2 + z^2$$



SPHERICAL COORDINATES!

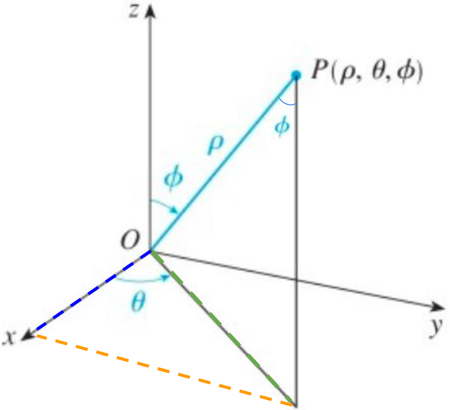
Sometimes the easiest conversion is into Spherical Coordinates, as follows

$$x = \rho \sin(\phi)\cos(\theta)$$

$$z = \rho \cos(\phi)$$

Note, that ρ is again the length, so we have an easy formula relating it to x,y,z:

$$\rho^2 = x^2 + y^2 + z^2$$



Spherical Coordinates!

Sometimes the easiest conversion is into Spherical Coordinates, as follows

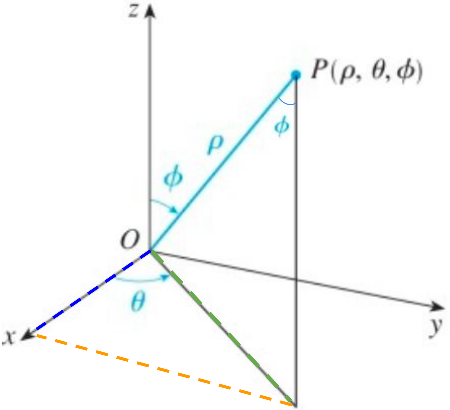
$$x = \rho \sin(\phi) \cos(\theta)$$

$$y = \rho \sin(\phi) \sin(\theta)$$

$$z = \rho \cos(\phi)$$

Note, that ρ is again the length, so we have an easy formula relating it to x,y,z:

$$\rho^2 = x^2 + y^2 + z^2$$



Spherical Coordinates!

What's so great about Spherical Coordinates?

Before a sphere was defined as:

$$x^2 + y^2 + z^2 = (\text{radius})^2$$

Now, we can define it as:

$$\rho = \text{radius}$$

Fixing ϕ

If we fix ϕ, we get interesting geometric shapes.

When $\phi = 0$, we get the positive z-axis.

When ϕ is between 0 and $\pi/2$, we get a cone opening from the origin in the positive z-direction.

When $\phi = \pi/2$, we get the xy-plane (a flat cone!).

When ϕ is between $\pi/2$ and π, we get a cone opening from the origin in the negative z-direction.

When $\phi = -\pi$, we have the negative z-axis.

This means we can usually don't have to go past the extreme values $\phi = 0$ and $\phi = \pi$.

Fixing θ

When we fix θ, we get a vertical half-plane from the z-axis in the θ direction. This is the same as in the case of cylindrical coordinates.

SPHERICAL COORDINATES!

Sometimes the easiest conversion is into Spherical Coordinates, and when changing a triple integral, on a region, R:

$$\rho_1 \leq \rho \leq \rho_2, \quad \theta_1 \leq \theta \leq \theta_2 \quad \phi_1 \leq \phi \leq \phi_2$$

$$\int\int_R\int f(x,y,z)\, dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\phi), \rho\cos(\phi))\rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta$$

Looks gross…. Sooooo great!!!

Spherical Coordinates!

$$\iiint_R f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\phi), \rho\cos(\phi))\rho^2\sin(\phi)\;d\rho\;d\phi\;d\theta$$

$$dV \rightsquigarrow \rho^2\sin(\phi)d\rho\;d\phi\;d\theta$$

Looks gross…. Sooooo great!!!

Spherical Coordinates!

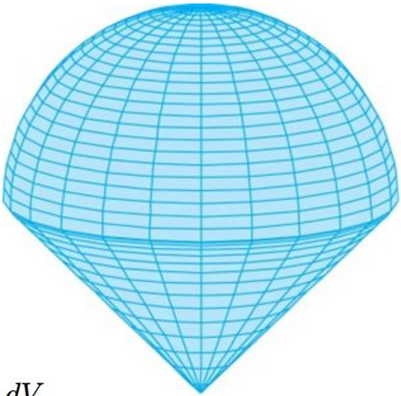
Example:

Find the volume of the "ice cream cone" cut from the solid sphere

$$x^2 + y^2 + z^2 = 9$$

And the upper part of the cone

$$z = \sqrt{x^2 + y^2}$$



RECALL: $\text{Volume of } R = \int\int_R\int \, dV$

Spherical Coordinates!

Example:

Find the volume of the "ice cream cone" cut from the solid sphere

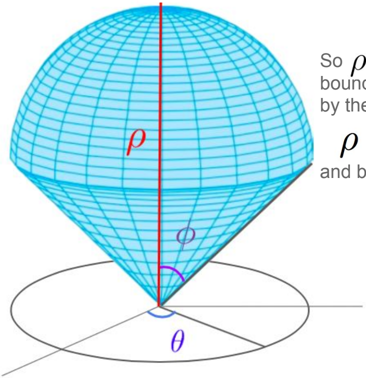
$$x^2 + y^2 + z^2 = 9$$

And the upper part of the cone

$$z = \sqrt{x^2 + y^2}$$

$$V = \int_?^? \int_?^? \int_0^3 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

My favorite way to find the bounds in this case is to analyze the picture...



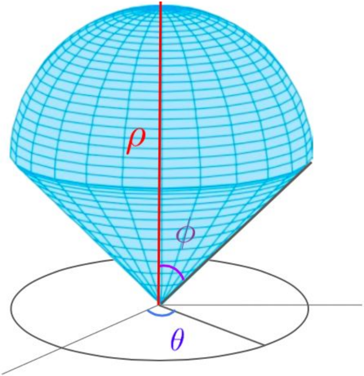
So ρ is bounded above by the sphere

$$\rho = 3$$

and below by 0

Spherical Coordinates!

Example:



Now, to find ϕ we need to find the intersection of the cone and the sphere

$$x^2 + y^2 + z^2 = 9$$

&

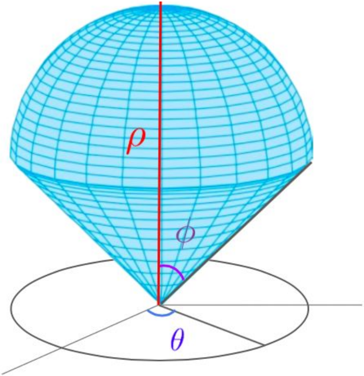
$$z = \sqrt{x^2 + y^2}$$

$$z^2 + z^2 = 2z^2 = 9$$

$$z = \frac{3}{\sqrt{2}}$$

Spherical Coordinates!

Example:



Now, to find ϕ we need to find the intersection of the cone and the sphere

$$z = \frac{3}{\sqrt{2}}$$

&

$$z = \rho\cos(\phi)$$

$$\frac{3}{\sqrt{2}} = 3\cos(\phi)$$

$$\phi = \frac{\pi}{4}$$

Spherical Coordinates!

Example:

Find the volume of the "ice cream cone" cut from the solid sphere

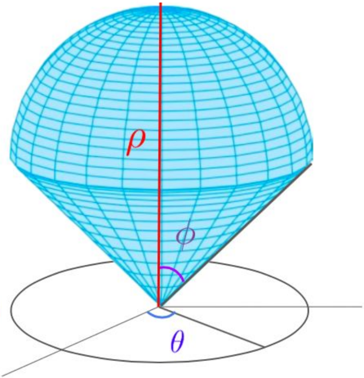
$$x^2 + y^2 + z^2 = 9$$

And the upper part of the cone

$$z = \sqrt{x^2 + y^2}$$

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$$

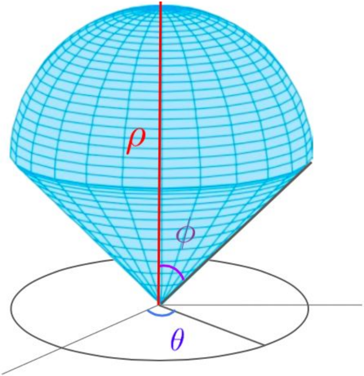
My favorite way to find the bounds in this case is to analyze the picture...



Spherical Coordinates!

Example:

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$



Now you Try!

Sketch the regions with following volumes, and solve for the volume:

- $$\int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^4 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$
- $$\int_0^{2\pi} \int_0^{\pi} \int_2^5 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$